A possible extension of lower and upper probabilities to the case of fuzzy random variables

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Starting point - Dempster's concept of interval-valued probabilities (1967)

- Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X : \Omega \rightarrow \mathbb{R}$ is a random variable that can not be observed precisely.
- Instead of X it is possible to observe the interval $\tilde{X}(\omega) = [\underline{X}(\omega), \overline{X}(\omega)]$, which contains the true value $X(\omega)$ for every $\omega \in \Omega$.
- The random variable induces a probability measure $\mathcal{P}^X$ on the Borel family $\mathcal{B}(\mathbb{R})$.
- **What can be said about the distribution $\mathcal{P}^X$ of X by looking at the random interval $\tilde{X}$?**

Starting point - Dempster's concept of interval-valued probabilities (1967)

- Assuming that $\tilde{X}$ is a random interval it is possible to calculate lower and upper bounds $\underline{\pi}(B)$ and $\overline{\pi}(B)$ for the probability $\mathcal{P}^X(B)$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned} \underline{\pi}(B) &:= \mathcal{P}\big(\{\omega \in \Omega : \tilde{X}(\omega) \subseteq B\}\big) \\ \overline{\pi}(B) &:= \mathcal{P}\big(\{\omega \in \Omega : \tilde{X}(\omega) \cap B \neq \emptyset\}\big) \end{aligned} \tag{1}$$

- Obviously $\underline{\pi}(B) \leq \mathcal{P}^X(B) \leq \overline{\pi}(B)$ holds for every Borel set $B \in \mathcal{B}(\mathbb{R})$.
- According to A. Dempster $\underline{\pi}$ and $\overline{\pi}$ are called *lower and upper probabilities induced by* $\tilde{X}$.

Starting point - Dempster's concept of interval-valued probabilities (1967)

- The same construction can also be done in the multidimensional setting - the random variable is replaced by a random vector and the containing interval is replaced by a containing random convex compact non-empty set.
- Regarding *fuzzy random variables as natural generalization of random intervals* it seems evident that the above concept can be applied directly - simply calculate lower and upper probabilities for every $\alpha \in (0, 1]$ and aggregate them in a suitable way.

Definition (Fuzzy vector)

A *d-dimensional fuzzy vector* $\xi^\star$ is a real-valued function on $\mathbb{R}^d$ with the following properties:

1. $0 \leq \xi^\star(x) \leq 1 \quad \forall x \in \mathbb{R}^d$
2. $\forall \alpha \in (0,1]$ the so-called α-cut $[\xi^\star]_\alpha$, defined by $[\xi^\star]_\alpha := \{x \in \mathbb{R}^d : \xi^\star(x) \geq \alpha\}$, is a non-empty compact convex set.

If $d = 1$ then $\xi^\star$ is called *fuzzy number*.

- For every d-dimensional fuzzy vector $\xi^\star$ the support $\mathrm{supp}(\xi^\star)$ is defined by

$$\mathrm{supp}(\xi^\star) := \mathrm{cls}\big(\{x \in \mathbb{R}^d : \xi^\star(x) > 0\}\big). \tag{2}$$

- $\mathcal{F}_c(\mathbb{R}^d)\dots$ d-dimensional fuzzy vectors
- $\mathcal{F}_{c,c}(\mathbb{R}^d)\dots$ d-dimensional fuzzy vectors with compact support
- $\xi^\star \oplus \eta^\star$ denotes the common sum of two d-dimensional fuzzy vectors (fuzzy numbers), i.e.

$$[\xi^\star \oplus \eta^\star]_\alpha = [\xi^\star]_\alpha + [\eta^\star]_\alpha = \{x + y : x \in [\xi^\star]_\alpha, y \in [\eta^\star]_\alpha\}$$

- The following two types of metrics on $\mathcal{F}_{c,c}(\mathbb{R}^d)$ will be used:

$$\delta^\star_{H,\infty}(\xi^\star, \eta^\star) := \sup_{\alpha\in(0,1]} \delta_H\big([\xi^\star]_\alpha, [\eta^\star]_\alpha\big) \tag{3}$$

$$\delta^\star_{H,p}(\xi^\star, \eta^\star) := \left(\int_{(0,1]} \big(\delta_H([\xi^\star]_\alpha, [\eta^\star]_\alpha)\big)^p d\alpha\right)^{1/p} \tag{4}$$

- If $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and $X^\star : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ $(d \geq 1)$ is a fuzzy-valued function on Ω, then the following notation will be used for every $\omega \in \Omega$ and every $\alpha \in (0, 1]$:

$$X_\alpha(\omega) := [X^\star(\omega)]_\alpha = \left\{x \in \mathbb{R}^d : \left(X^\star(\omega)\right)(x) \geq \alpha\right\} \quad (5)$$

Definition (Fuzzy random vector)

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, then a function $X^\star : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is called *(d-dimensional) fuzzy random vector* if

$$\left\{\omega \in \Omega : X_\alpha(\omega) \cap B \neq \emptyset\right\} \in \mathcal{A} \quad (6)$$

holds for every $B \in \mathcal{B}(\mathbb{R}^d)$ and every $\alpha \in (0, 1]$. If $d = 1$, then $X^\star$ is called *fuzzy random variable*.

(1) Fuzzy relative frequencies - Random set approach

- Given $A \subseteq \mathbb{R}^d$ and a d-dimensional fuzzy sample $x_1^\star, x_2^\star, \ldots, x_n^\star$.
- For $\alpha \in (0,1]$ define

$$\begin{aligned} \underline{h}_{n,\alpha}(A) &:= \frac{1}{n}\#\Big\{ i \in \{1,\ldots,n\} : [x_i^\star]_\alpha \subseteq A \Big\} && \text{(contained)} \\ \overline{h}_{n,\alpha}(A) &:= \frac{1}{n}\#\Big\{ i \in \{1,\ldots,n\} : [x_i^\star]_\alpha \cap A \neq \emptyset \Big\} && \text{(hitting).} \end{aligned}$$

- This yields a nested family of closed intervals

$$h_{n,\alpha}(A) := [\underline{h}_{n,\alpha}(A), \overline{h}_{n,\alpha}(A)] \quad \text{for every } \alpha \in (0,1],$$

that in turn induces a unique fuzzy number (the *convex hull* of the family $h_{n,\alpha}(A)$), which will be denoted by $h_n^\star(A)$.

Figure: Fuzzy sample of size 10

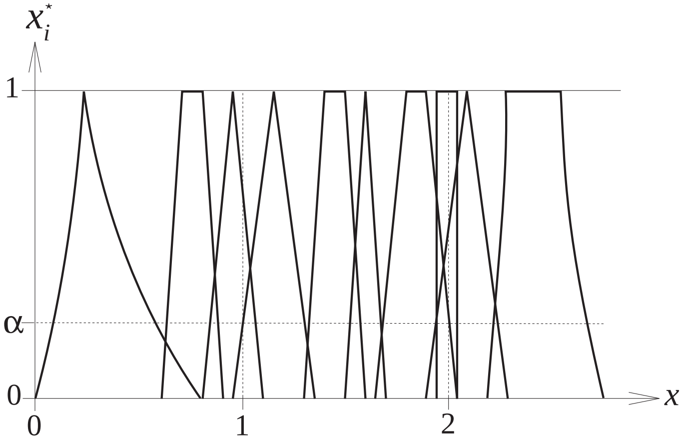
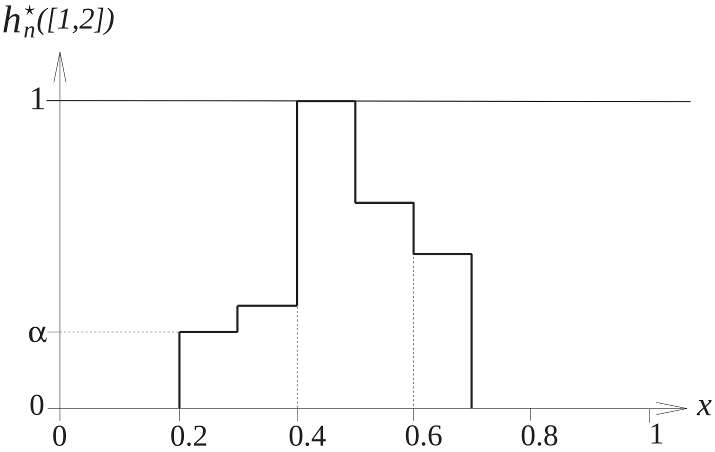


Figure: Fuzzy relative frequency $h_n^\star([1,2])$



Lemma (Basic properties of $h_n^\star(\cdot)$ as set-mapping)

- $\underline{h}_{n,\alpha}(\mathbb{R}^d) = \overline{h}_{n,\alpha}(\mathbb{R}^d) = 1 \quad \forall \alpha \in (0,1]$
- $\underline{h}_{n,\alpha}(\emptyset) = \overline{h}_{n,\alpha}(\emptyset) = 0, \quad \forall \alpha \in (0,1]$
- $\underline{h}_{n,\alpha}(A), \overline{h}_{n,\alpha}(A) \in [0,1] \quad \forall A \subseteq \mathbb{R}^d, \alpha \in (0,1]$
- $\underline{h}_{n,\alpha}(\cdot)$ *and* $\overline{h}_{n,\alpha}(\cdot)$ *are monotonic set functions for every* $\alpha \in (0,1]$
- $\underline{h}_{n,\alpha}(\cdot)$ *is a superadditive and* $\overline{h}_{n,\alpha}(\cdot)$ *is a subadditive set function for every* $\alpha \in (0,1]$
- $\ldots$

Fuzzy probability distributions

- Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X^\star : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is a d-dimensional fuzzy random vector.
- $X^\star$ induces families $(\underline{\pi}_\alpha)_{\alpha\in(0,1]}$ and $(\overline{\pi}_\alpha)_{\alpha\in(0,1]}$ of upper and lower probabilities on $\mathcal{B}(\mathbb{R}^d)$ (in Dempster's sense) by

$$\begin{aligned} \underline{\pi}_\alpha(A) &:= \mathcal{P}\big(\{\omega \in \Omega : X_\alpha(\omega) \subseteq A\}\big) \\ \overline{\pi}_\alpha(A) &:= \mathcal{P}\big(\{\omega \in \Omega : X_\alpha(\omega) \cap A \neq \emptyset\}\big). \end{aligned} \tag{7}$$

- These families can be aggregated to a unique fuzzy number $\mathbb{P}^\star(A) \in \mathcal{F}_{c,c}(\mathbb{R})$ (again the convex hull) called *fuzzy probability of the set A*.
  The $\mathcal{F}_{c,c}(\mathbb{R})$-valued set function $\mathbb{P}^\star(\cdot)$, defined on $\mathcal{B}(\mathbb{R}^d)$, is called *fuzzy probability distribution induced by $X^\star$*.

Lemma

Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X^\star : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is a d-dimensional fuzzy random vector. Then the upper and lower probabilities $\underline{\pi}_\alpha$ and $\overline{\pi}_\alpha$ fulfill the following properties for every $\alpha \in (0,1]$

- $\underline{\pi}_\alpha(\mathbb{R}^d) = \overline{\pi}_\alpha(\mathbb{R}^d) = 1 \quad \underline{\pi}_\alpha(\emptyset) = \overline{\pi}_\alpha(\emptyset) = 0$
- $\underline{\pi}_\alpha(A^c) = 1 - \overline{\pi}_\alpha(A)$ *holds for every* $A \in \mathcal{B}(\mathbb{R}^d)$
- $\underline{\pi}_\alpha(\cdot)$ *is a superadditive and* $\overline{\pi}_\alpha(\cdot)$ *is a subadditive set function*
- $\underline{\pi}_\alpha(\cdot)$ *and* $\overline{\pi}_\alpha(\cdot)$ *are monotonic set functions*
- $\overline{\pi}_\alpha(\cdot)$ *is a completely alternating, $\mathcal{C}$-convex Choquet capacity on* $\mathcal{K}_c(\mathbb{R}^d)$

- The precise definition of $\mathbb{P}^\star(A) \in \mathcal{F}_{c,c}(\mathbb{R})$ is

$$\big(\mathbb{P}^\star(A)\big)(x) := \sup\left\{\alpha \in (0,1] : x \in ([\underline{\pi}_\alpha(A), \overline{\pi}_\alpha(A)]\right\}. \qquad (8)$$

- This is easily seen to be equivalent to setting

$$[\mathbb{P}^\star(A)]_\alpha = [\underline{p}_\alpha(A), \overline{p}_\alpha(A)] = \bigcap_{\beta<\alpha} [\underline{\pi}_\beta(A), \overline{\pi}_\beta(A)]. \qquad (9)$$

- Consequently all the properties stated in Lemma 4 also hold for the α-cuts of $\mathbb{P}^\star$.

Motivation

- **Well known:** Given a sequence of pairwise independent identically distributed integrable random variables $(X_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{A}, \mathcal{P})$ the classical relative frequency $h_n(A,\cdot)$ converges with probability 1 for $n \longrightarrow \infty$ to $\mathcal{P}(X \in A)$.
- **Question:** Does a similar result also hold for fuzzy relative frequencies?
  - I.e. if $(X_n^\star)_{n\in\mathbb{N}}$ is a sequence of pairwise independent identically distributed fuzzy random vectors on $(\Omega, \mathcal{A}, \mathcal{P})$ does the fuzzy relative frequency $h_n^\star(A,\cdot)$ converge for $n \longrightarrow \infty$?
  - If yes, in which sense (with respect to which metric?)

- Given a sequence $(X_n^\star)_{n\in\mathbb{N}}$ of d-dimensional fuzzy random vectors on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ define the fuzzy relative frequency $h_n^\star(A,\omega)$ of the Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ with respect to the sample $(X_n^\star(\omega))_{n\in\mathbb{N}}$ analogous to before as the convex hull of the family $\left[\underline{h}_{n,\alpha}(A,\omega), \overline{h}_{n,\alpha}(A,\omega\right]$, with

$$\begin{aligned}\underline{h}_{n,\alpha}(A,\omega) &:= \frac{1}{n}\#\Big\{i \in \{1,\ldots,n\} : [X_i^\star(\omega)]_\alpha \subseteq A\Big\}\\ \overline{h}_{n,\alpha}(A,\omega) &:= \frac{1}{n}\#\Big\{i \in \{1,\ldots,n\} : [X_i^\star(\omega)]_\alpha \cap A \neq \emptyset\Big\}.\end{aligned}$$

- $X^\star$ and $Y^\star$ are called *independent*, iff the following equality holds for every $\alpha \in (0,1]$ and all Borel sets $B_1, B_2 \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{P}(X_\alpha \subseteq B_1, Y_\alpha \subseteq B_2) = \mathcal{P}(X_\alpha \subseteq B_1)\,\mathcal{P}(Y_\alpha \subseteq B_2)$$

Theorem (Convergence of fuzzy relative frequencies)

Suppose that $X^\star, X_1^\star, X_2^\star, \cdots$ is a sequence of pairwise independent, identically distributed d-dimensional fuzzy random vectors and that $A \in \mathcal{B}(\mathbb{R}^d)$ is an arbitrary Borel set.
Then with probability 1 the fuzzy relative frequency $h_n^\star(A,\cdot)$ converges to $\mathbb{P}^\star(A)$ with respect to the metric $\delta^\star_{H,\infty}$ and with respect to the metric $\delta^\star_{H,p}$ for every $p \geq 1$.

Corollary (Consistency)

Under the above assumptions the fuzzy relative frequency $h_n^\star(A,\cdot)$ is a strongly consistent estimator for the fuzzy-valued probability $\mathbb{P}^\star(A)$.

Open points

- Is it possible to prove a Glivenko-Cantelli-like Theorem in the following way (same setting as before with $d = 1$):

$$\sup_{x \in \mathbb{R}} D\Big(h_n^\star((-\infty, x]) \,,\, \mathbb{P}^\star((-\infty, x])\Big) \to 0 \qquad \text{a.s.}$$

- The presented concept of fuzzy relative frequencies has some disadvantages - in general a sample can not be reconstructed by looking at the fuzzy empirical distribution function $F_n^\star(x) := h_n^\star((-\infty, x])$.

Thank you for your attention !